Quantitative Methods – Applied Mathematics

Winter term 2023/24

Introduction / Revision

Prof. Dr. Frank Altrock
Email: f.altrock@hochschule-trier.de

Study board International Business

Semester							
7	Practical Project (abroad or within an international context)				Final Thesis		
6	Electives (abroad / any language is allowed)						
5	Electives (abroad / any language is allowed)						
4	Intercultural Management	Corporate Finance 2	Managerial Decision Making	Integrated Business Game	Seminar (building on Scientific Writing)		Languages
3	Strategic Managemet	Corporate Finance 1	Financial Accounting	Entrepreneurship	Scientific Writing	Quantitative Methods 3 (Applied Data Science)	Languages
2	Organization and HR Management	Operations Management	Managerial Accounting	International Economics	Spreadsheet Applications in Business	Quantitaive Methods 2 (Applied Statistics)	Languages
1	Introduction to Management	Marketing	Fundamentals of Accounting	Principles of Economics	Principles of Law	Quantitative Methods 1 (Applied Mathematics)	Languages
	Propaedeutics						

Applied Mathematics as a Mandatory basic module in International Business Study program

Program overview of the Applied Mathematics course

I. Differentiation

II. Univariate and Multivariate Optimization

III. Integration

IV. Matrix Algebra

V. Probability theory and combinatorics

VI. Random variables and probability distributions

General remark

Important!

The following slides summarize certain contents - especially calculation rules - which are assumed to be known.

The following exercises are intended for independent revision. Suggested solutions are provided online. In addition, relevant exercises have been compiled for you on MyMathLab under the title "Placement quiz/revision".

You may want to consult the textbook for deepening of your understanding of that content that will be assumed in the lecture QM1. This textbook is K. Sydsaeter/P. Hammond/ A. Strom/A. Carvajal: Essential Mathematics for Economic Analysis, 5th Edition, Pearson 2016. You can easily identify the relevant chapters.

Naming conventions for numbers (selection)

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}\backslash\mathbb{Q}$	Set of irrational numbers

Example, natural numbers: The non-negative integers used for counting. (Zero can therefore be counted among them) The totality of these numbers is called the set N of natural numbers.

$\mathbb{N} = \{0,1,2,3,4,\ldots\}$

Exercise 1

What is the characteristic property of irrational numbers? Which irrational numbers do you know?

Mathematics – introduction/revision

We use four basic arithmetic operations:

- Addition $+$
- Subtraction $-$
- Multiplication $*$
- Division $/$

For each arithmetic operation, there is a **neutral element** and an **inverse element**.

Neutral Element: Each element is reflected onto itself by the combination with a neutral element.
E.g. Addition: 5 + <u>0</u> = 5
E.g. Multiplication: 5 * <u>1</u> = 5

Inverse: If you use a calculation operation to combine an element and its inverse element, you get the neutral element as the result.
E.g. Addition: 5 + <u>(-5)</u> = 0
E.g. Multiplication: 5 * <u>1/5</u> = 1

Exercise 2

Let $a = 7$.

a) What is the corresponding neutral element of addition?
b) What is the corresponding neutral element of multiplication?
c) What is the corresponding inverse of addition?
d) What is the corresponding inverse of multiplication?

Basic laws of arrangement (excerpt)

– Between two natural numbers a and b there is exactly one of the following relations:
$a < b$, $a = b$, $a > b$.

– $a = a$ (Reflexivity)

– From $a = b$ follows $b = a$ (symmetry)

– From $a = b$ and $b = c$ follows $a = c$ (transitivity).

Basic laws of addition (excerpt)

– For two natural numbers a and b, there always exists the sum $a + b$ in the range of natural numbers.

– From $a = a'$ and $b = b'$ follows $a + b = a' + b'$ (Uniqueness).

– $a + b = b + a$ (**Commutative law)**

– $(a + b) + c = a + (b + c)$ (**Associative law**)

– From $a < b$ follows $a + c < b + c$ (**Monotonicity law**).

Basic laws of subtraction (Excerpt)

– If for two natural numbers a and b there exists a natural number x that satisfies the equation $a + x = b$, then $x = b - a$ is called the **difference** of b and a.

– $x = b - a$ is uniquely determined.

Basic laws of multiplication (Excerpt)

– For two natural numbers a and b, the product $a \cdot b$ always exists in the range of natural numbers. „$a \cdot b$“ is also written as „ab“.

– From $a = a'$ and $b = b'$ follows $a \cdot b = a' \cdot b'$ (Uniqueness).

– $a \cdot b = b \cdot a$ (**Commutative law**)

–$(a \cdot b)c = a(b \cdot c)$ (**Associative law**)

– $(a + b)c = a \cdot c + b \cdot c$ (**Distributive law**)

– From $a < b$ and $c > 0$ follows $a \cdot c < b \cdot c$ (**Monotonicity law**).

Basic laws of division

– If for two natural numbers a and b, where $\boldsymbol{a \neq 0}$, there exists a natural number x that satisfies the equation $ax = b$, then $x = b / a$ (quotient of b and a) also holds.

– The basic laws of division can be supplemented for the rational numbers by: For every two rational numbers $a \neq 0$ and b there exists exactly one rational number that satisfies the equation $ax = b$.

Selected calculation rules

– Factoring out: $4 + 8 = 2 \cdot (2 + 4)$

– Reducing fractions:

$$\frac{\cancel{a} \cdot b}{\cancel{a} \cdot c} = \frac{b}{c} \rightarrow \underline{\text{Example:}}\ \frac{3 \cdot 5}{3 \cdot 7} = \frac{5}{7}$$

but $\frac{a+b}{a+c} \neq \frac{b}{c}$!!

Furthermore: $\frac{a \cdot b + a \cdot d}{a \cdot c} = \frac{\cancel{a} \cdot b}{\cancel{a} \cdot c} + \frac{\cancel{a} \cdot d}{\cancel{a} \cdot c} = \frac{b + d}{c} \rightarrow$ Example: $\frac{3 \cdot 5 + 3 \cdot 4}{3 \cdot 7} = \frac{5 + 4}{7} = \frac{9}{7}$

Selected calculation rules

– Adding up fractions:

$$\frac{a}{c}+\frac{b}{d}=\frac{a}{c}\cdot\mathbf{1}+\frac{b}{d}\cdot\mathbf{1}=\frac{a}{c}\cdot\underbrace{\frac{d}{d}}_{=1}+\frac{b}{d}\cdot\underbrace{\frac{c}{c}}_{=1}=\frac{a\cdot d+b\cdot c}{c\cdot d}$$

→ Example: $\frac{3}{5}+\frac{2}{7}=\frac{3}{5}\cdot\frac{7}{7}+\frac{2}{7}\cdot\frac{5}{5}=\frac{3\cdot 7+2\cdot 5}{5\cdot 7}=\frac{31}{35}$

– Multiplication of fractions:

$$\frac{a}{c}\cdot\frac{b}{d}=\frac{a\cdot b}{c\cdot d}$$ → Example: $\frac{3}{5}\cdot\frac{2}{7}=\frac{6}{35}$

– Division of fractions:

$$\frac{\frac{a}{c}}{\frac{b}{d}}=\frac{a}{c}\cdot\frac{d}{b}$$ (Invert value) → Example: $\frac{\frac{3}{5}}{\frac{2}{7}}=\frac{3}{5}\cdot\frac{7}{2}=\frac{21}{10}$

Exercise 3

Calculate the following:

$a)\ \frac{1}{4}+\frac{5}{7}$

$b)\ \frac{2}{3}\cdot\frac{3}{7}$

$c)\ \frac{\frac{5}{9}}{\frac{2}{3}}$

When multiplying two bracket expressions, multiply each element of one bracket by each element of the other bracket, following the sign rule:

$$(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{c}-\boldsymbol{d})=\boldsymbol{a}\cdot\boldsymbol{c}-\boldsymbol{a}\cdot\boldsymbol{d}+\boldsymbol{b}\cdot\boldsymbol{c}-\boldsymbol{b}\cdot\boldsymbol{d}$$

→ Example: $(3x+4y)\cdot(z-x^2)=3xz-3x^3+4yz-4yx^2$

When multiplying more than two bracket expressions, apply this procedure step-by-step:

$$(\mathrm{a}+\mathrm{b})\cdot(\mathrm{c}-\mathrm{d})\cdot(\mathrm{e}-\mathrm{f}-\mathrm{g})=(\mathrm{a}\cdot\mathrm{c}-\mathrm{a}\cdot\mathrm{d}+\mathrm{b}\cdot\mathrm{c}-\mathrm{b}\cdot\mathrm{d})\cdot(\mathrm{e}-\mathrm{f}-\mathrm{g})$$

$$=\mathrm{ace}-\mathrm{acf}-\mathrm{acg}-\mathrm{ade}+\mathrm{adf}+\mathrm{adg}+\mathrm{bce}-\mathrm{bcf}-\mathrm{bcg}-\mathrm{bde}+\mathrm{bdf}+\mathrm{bdg}$$

Exercise 4

Solve the following:

$(5zx-4z^3)\cdot(22z+3x^2)$

As a special case, one obtains the well-known **binomial formulas**:

$$\boldsymbol{(a+b)^2 = a^2 + 2ab + b^2}$$

$$\boldsymbol{(a-b)^2 = a^2 - 2ab + b^2}$$

$$\boldsymbol{(a+b)(a-b) = a^2 - b^2}$$

→ Examples:

$$(2a+3b)^2 = 4a^2 + 12ab + 9b^2$$

$$(ax-by)^2 = a^2x^2 - 2abxy + b^2y^2$$

$$\left(4u+\sqrt{2}v\right)\left(4u-\sqrt{2}v\right) = 16u^2 - 2v^2$$

Powers

The n-th power of any real number x is defined as the n-fold product of x with itself:

$$x \cdot x = x^2 \qquad x \cdot x \cdot x = x^3 \qquad \rightarrow \qquad \boldsymbol{\overbrace{x \cdot x \cdot \dots \cdot x \cdot x}^{\text{n times}} = x^n}$$

Where x is the **base** and n = 1, 2, 3, . . . is the **exponent**.

Furthermore, the following applies to powers:

$\boldsymbol{x^m \cdot x^n = x^{m+n}}$ → Example: $x^5 \cdot x^3 = x^8$ or $x^{-5} \cdot x^3 = x^{-2}$

$\boldsymbol{\frac{x^m}{x^n} = x^{m-n}}$ → Examples: $\frac{x^5}{x^3} = x^2$ or $\frac{x^3}{x^5} = x^{-2}$ or $\frac{x^5}{x^5} = x^0 = 1$

$\boldsymbol{(x^m)^n = x^{m \cdot n}}$ → Example: $(x^3)^5 = x^{15}$

$\boldsymbol{(x \cdot y)^n = x^n \cdot y^n}$ → Example: $(x \cdot y)^5 = x^5 \cdot y^5$

Moreover:

The following transformations are especially interesting for derivatives, e.g. to apply the power rule:

$\boldsymbol{\frac{a}{x^y} = ax^{-y}}$ → Example: $\frac{1}{x^2} = x^{-2}$ or $\frac{5}{x^3} = 5x^{-3}$

$\boldsymbol{\sqrt[a]{x^y} = x^{\frac{y}{a}}}$ → Example: $\sqrt[3]{x^4} = x^{\frac{4}{3}}$

Exercise 5

Show as a power:

a) $\sqrt[13]{x^5}$

b) $5 \cdot \sqrt[3]{k^7}$

c) $\frac{1}{b^7}$

d) $\frac{21}{x^{17}}$

Combine:

a) $y^3 \cdot y^4$

b) $\frac{x^8}{x^5}$

c) $(p^3)^{2,5}$

Logarithim:

The logarithm c of a positive real number a to a positive real base b other than one is the real number c by which the base b must be raised to the power of a to obtain a:

$$\mathbf{log}_{b}\,\boldsymbol{a} = \boldsymbol{c} \qquad \rightarrow \qquad \boldsymbol{b}^{\boldsymbol{c}} = \boldsymbol{a}$$

<u>Example</u>:

$$\log_2 8 = 3 \qquad \rightarrow \qquad 2^3 = 8$$

Calculators usually distinguish between the logarithm of 10 (log) and the logarithm to base e (ln), which is also called “natural logarithm". Some models also offer individual logarithm calculations with log[redacted]".

The following calculation rules apply to logarithms:

$\mathbf{log}(\boldsymbol{x} \cdot \boldsymbol{y}) = \mathbf{log}(\boldsymbol{x}) + \mathbf{log}(\boldsymbol{y})$ → <u>Example:</u> $\log(3 \cdot 5) = \log(3) + \log(5)$

$\mathbf{log}\left(\frac{\boldsymbol{x}}{\boldsymbol{y}}\right) = \mathbf{log}(\boldsymbol{x}) - \mathbf{log}(\boldsymbol{y})$ → <u>Example:</u> $\log\left(\frac{3}{5}\right) = \log(3) - \log(5)$

$\mathbf{log}(\boldsymbol{x}^{\boldsymbol{y}}) = \boldsymbol{y} \cdot \mathbf{log}(\boldsymbol{x})$ → <u>Example:</u> $\log(5^3) = 3 \cdot \log(5)$

Exercise 6

Calculate the following:

a) $\ln(3)$
b) $\log(120)$
c) $\log_2(33)$

First combine and then calculate:

a) $\ln(4) + \ln(3)$
b) $\log(3^7)$
c) $\ln\left(\frac{2}{3}\right)$

Linear equations with one unknown (variable)

- The basic form of the linear equation with one unknown x is as follows

$$ax = b$$

- Here a, b are real numbers. Solving the equation means specifying all real numbers that satisfy the equality condition when substituted for $\boldsymbol{x}$.
- Since division by $a \neq 0$ is equivalent to multiplication by $1/a$, the only possible solution of $ax = b$

$$x = \frac{b}{a} \qquad where \qquad a \neq 0$$

- If, on the other hand, $a = 0$, two cases can be observed
  - $b \neq 0$, then the equation becomes $0 \cdot x = b$, which is a contradiction on ist own. In this case, **no solution** exists.
  - $b = 0$, then the equation becomes $0 \cdot x = 0$. This equation is satisfied for all real numbers. In this case, $\boldsymbol{x}$ is any number.
- So possible: unique solution, no solution, infinitely many solutions.

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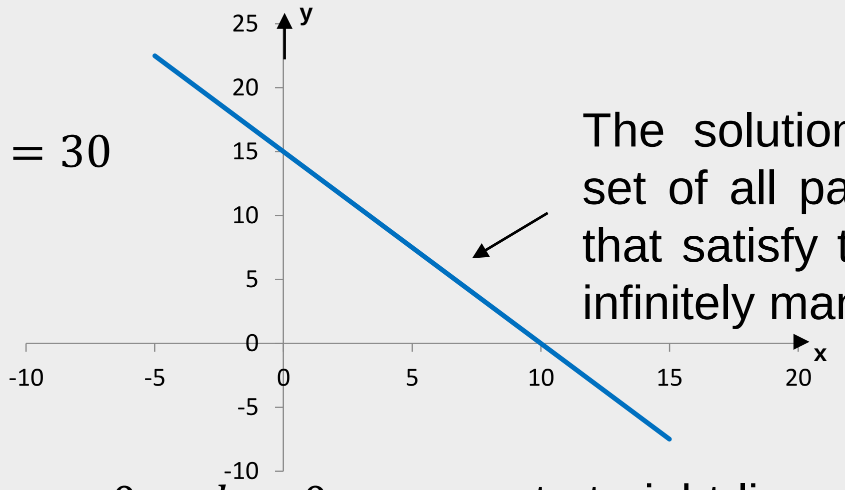
Linear equations with two unknowns (variables)

Such an equation can be expressed as

$$\boldsymbol{ax + by = c}$$

Here the unknown values x and y can be any real numbers (variables) associated in the given way. The equation also represents a linear function equation whose graph is a straight line in the x, y - coordinate system.

E.g.:
$3x + 2y = 30$



The solution is obviously the set of all pairs of values (x,y) that satisfy the equation. Here infinitely many.

The special cases where $a = 0$ or $b = 0$ represent straight lines that are parallel to the y or x - axis, respectively.

(Graph is made using MS Excel)

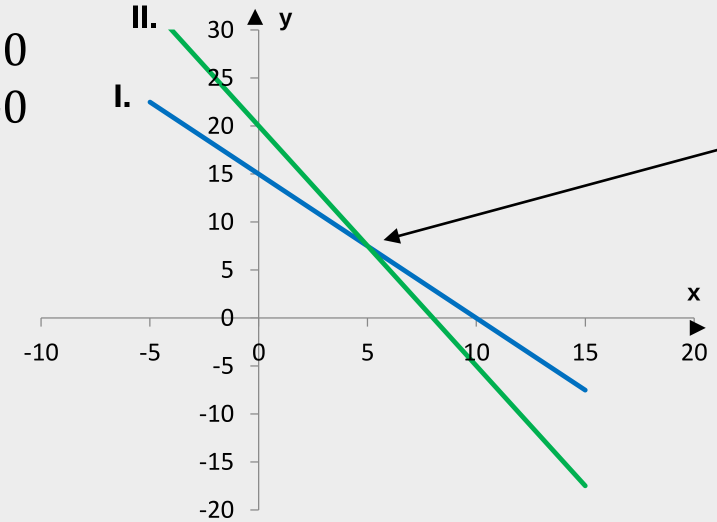
Linear equations with two unknowns (variables)

A set of two equations with two unknowns has the following general configuration:

$$\text{I.}\quad \boldsymbol{a_1 x + b_1 y = c_1}$$
$$\text{II.}\quad \boldsymbol{a_2 x + b_2 y = c_2}$$

These equations can each be plotted as straight lines in the x,y - coordinate system. If the straight lines intersect at a point, the coordinates of this intersection are the only solution of the given set of equations.

E.g.:
I. $3x + 2y = 30$
II. $5x + 2y = 40$



Unique solution of the set of equations at

$x = 5$ and $y = 7{,}5$.

(Graph is made using MS Excel)

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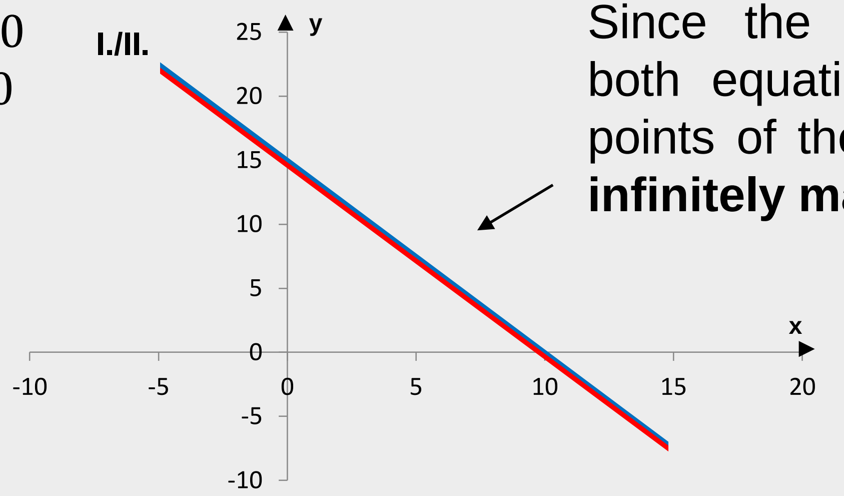
Linear equations with two unknowns (variables)

If both equations contain the same information, there are infinitely many solutions, because one equation is only a multiplier of another one. For the solution of sets of equations, therefore, one always needs as many independent equations as there are unknowns.

E.g.:

I. $3x + 2y = 30$

II. $6x + 4y = 60$



Since the equations are identical, both equations are satisfied on all points of the straight line: There are **infinitely many solutions**.

(Graph is made using MS Excel)

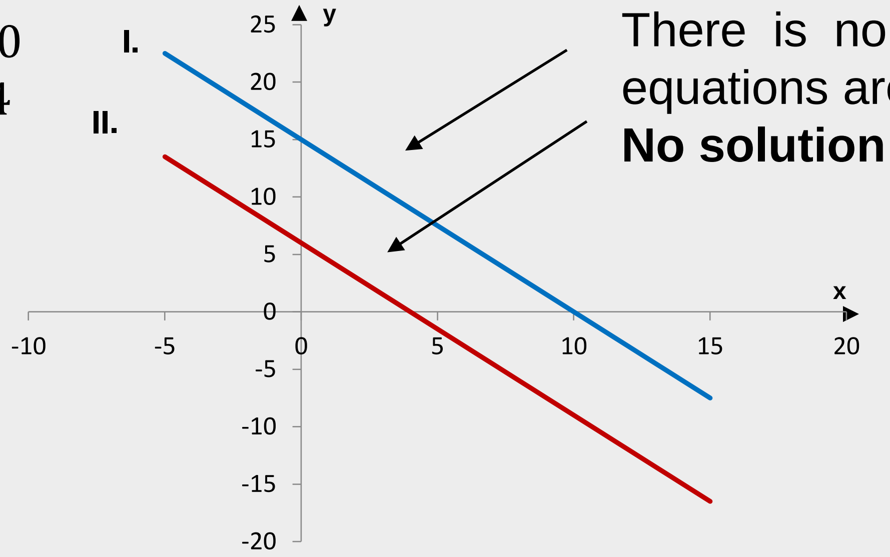
Linear equations with two unknowns (variables)

The case where no solution exists occurs when the straight lines represented by I and II are parallel but do not intersect.

E.g.:

I. $3x + 2y = 30$

II. $6x + 4y = 24$



There is no point where both equations are fulfilled:
No solution!

To sum up, even for the equations with two unknowns the following can be true:

- **Unique solution**
- **Infinitely many solutions**
- **No Solution**

(Graph is made using MS Excel)

Linear equations with two unknowns (variables)

To solve the equations with two unknowns, you can use the following methods:

- **Addition**
- **Substitution**
- **Equating**

Addition Method

One adds a certain (possibly also negative) factor of the II. equation to the I. equation (or vice versa) in such a way that an unknown no longer exists.

Example:

I. $\mathbf{2\,x + 3y = \ 4}$ $\quad / -2 \cdot \mathrm{II}$

II. $\mathbf{x - 2y = -5}$

I. $\mathbf{7y = 14}$

I. $\underline{\underline{\mathbf{y = \ 2}}}$

Subtract the second equation 2 times from the first. So that the x "disappears".

Unique Solution!

Then substitute $y = 2$ in II. equation: $x - 2 \cdot 2 = -5 \rightarrow \underline{\underline{\boldsymbol{x = -1}}}$

Addition Method

Another example:

$$\begin{array}{lrcl} \text{I.} & 2x + 3y = & 4 & / -0{,}5 \cdot \text{II} \\ \text{II.} & 4x + 6y = & 8 & \\ \hline \text{I.} & 0x + 0y = & 0 & \end{array}$$

Infinitely many solutions!
(Equation I. and II. are the same)

Another example:

$$\begin{array}{lrcl} \text{I.} & 2x + 3y = & 4 & / -0{,}5 \cdot \text{II} \\ \text{II.} & 4x + 6y = & 10 & \\ \hline \text{I.} & 0x + 0y = & -1 & \end{array}$$

Contradiction, no solution!

Substitution method

Solve one of the equations for one unknown (e.g. y) and substitute the result into the other equation. Then you get an equation with only one unknown (e.g. x). Example with unique solution:

I. $2x + 3y = 4$

II. $x - 2y = -5 \quad / -2 \cdot y$

II‘. $x = 2y - 5$

Solve for x

→ Substitute in the I. equation:

$$2(2y-5) + 3y = 4$$
$$4y - 10 + 3y = 4$$
$$7y = 14$$
$$\underline{\underline{y = 2}}$$

→ Substitue in II‘.: $x = 2 \cdot 2 - 5 \rightarrow \underline{\underline{x = -1}}$

Equating method

Solve both equations for the same unknown (e.g., y) and set them equal, obtaining an equation with one unknown (e.g., x).

Example with unique solution:

I. $2x + 3y = 4$
II. $x - 2y = -5$

Solve both equations for the same variable, here for x.

I‘. $x = 2 - 1{,}5y$
II‘. $x = 2y - 5$

→ Equate:

$$2 - 1{,}5y = 2y - 5$$
$$7 = 3{,}5y$$
$$\underline{\underline{y = 2}}$$

→ substitute in II‘.: $x = 2 \cdot 2 - 5$ → $\underline{\underline{x = -1}}$

Exercise 7

Solve the following sets of equations using a method of your choice:

a) $$3x + y = 6$$
$$x - y = 10$$

b) $$2m - u = 6$$
$$-m - 2u = 3$$

c) $$5(x_2 + 2) - 3(x_1 + 1) = 23$$
$$3(x_2 - 2) = 19 - 5(x_1 - 1)$$

Quadratic equations (second-degree polynomial equations)

- The general expression of a quadratic equation with one unknown x is:

$$\boldsymbol{ax^2 + bx + c = 0}$$

- There are different approaches to solving a quadratic equation (p,q-formula, a,b,c-formula, quadratic completion). At this point, we will only consider the p,q formula. For this purpose, the general expression of the quadratic equation given above has to be converted into its normal form by dividing by the coefficient a:

$$\boldsymbol{x^2 + \underbrace{\frac{b}{a}}_{p}x + \underbrace{\frac{c}{a}}_{q} = 0}$$

- Now the p,q formula can be applied, afterwards two values for $\boldsymbol{x}$ are obtained:

$$\boldsymbol{x_{1/2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}}$$

Example p,q-formula:

$2x^2 - 12x + 15 = -1 \qquad /+1$

$2x^2 - 12x + 16 = 0 \qquad /:2$

$x^2 - 6x + 8 = 0$

↑ Normal form

→ Then follows:

$$x_{1/2} = -\frac{-6}{2} \pm \sqrt{\left(\frac{-6}{2}\right)^2 - 8}$$

$$\underline{\underline{\boldsymbol{x_1 = 4}}}$$

$$\underline{\underline{\boldsymbol{x_2 = 2}}}$$

Exercise 8

Solve the following equations:

a) $(x - 6)(x + 5) = 0$

b) $5k^2 = 125k$

c) $3x^2 - 27 = 0$

Sum and product symbols

Sum and product symbols are used for the simplified representation of sums and products. They are often used in complex formulas, e.g. in statistical formulas.

- Summation symbol:

$$\sum_{i=1}^{n} x$$

It reads: „the sum of n first whole numbers “. The index always increases by +1 when summing up.

- Product symbol:

$$\prod_{i=1}^{n} x$$

- Factorial:
  $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Exercise 9

1) Determine the following:

a) $\sum_{i=1}^{10} 2$

b) $\sum_{i=1}^{5} x$

c) $\sum_{i=1}^{7} i$

d) $\sum_{i=2}^{6} x^i$

e) $\sum_{i=1}^{10} (2 \cdot i - i^2)$

f) $\sum_{k=1}^{5} i$

2) Find the sum of all numbers from 1 to 100.